Risk and Decision Analysis in Civil Engineering
M.J. Kallen and S.P. Kuniewski (Eds.)
IOS Press

Modelling track geometry by a bivariate Gamma wear process, with application to maintenance

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Abstract. This paper discusses the maintenance optimization of a railway track, based on the observation of two dependent randomly increasing deterioration indicators. These two indicators are modelled through a bivariate Gamma process constructed by trivariate reduction. Empirical and maximum likelihood estimators are given for the process parameters and tested on simulated data. The EM algorithm is used to compute the maximum likelihood estimators. A bivariate Gamma process is then fitted to real data of railway track deterioration. Preventive maintenance scheduling is studied, ensuring that the railway track keeps a good quality with a high probability. The results are compared to those based on both indicators taken separately, and also on one single indicator (usually taken for current track maintenance). The results based on the joined information are proved to be safer than the other ones, which shows the interest of the bivariate model.

1 INTRODUCTION

This paper is concerned with the maintenance optimization of a railway track, based on the observation of two dependent randomly increasing deterioration indicators. The railway track is considered as deteriorated when any of these two indicators is beyond a given threshold. The point of the paper is the study of preventive maintenance scheduling, which must ensure that, given some observations provided by inspection, the railway track will remain serviceable until the next maintenance action with a high probability.

Track maintenance is a very expensive task to accomplish. Consequently, it is essential to carry out maintenance actions in an optimal way, while taking into account many parameters: safety and comfort levels to be guaranteed, available logistic means, ... The earlier the deterioration is detected,

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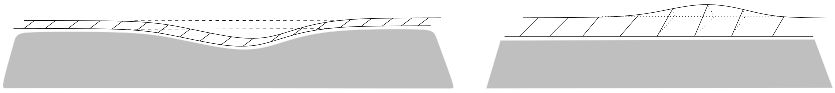

FIGURE 1. Levelling defects

the easier it is to schedule maintenance actions. The objective is therefore to develop a good prediction model.

Deterioration of track geometry is characterized by the development of different representative parameters like, for example, the levelling of the track. Figure 1 shows the defects that are measured by two of these parameters: the longitudinal (NL) and transversal (NT) levelling indicators.

At the SNCF (French National Railways), track inspections are programmed annually on a national level. The interval between two inspections on high speed tracks is currently about two weeks, the inspections are carried out by a modified high-speed train. The collected time series are transformed into indicators that sum up the state of the track over each km. These new indicators are referred to as synthesized Mauzin data. Numeric Mauzin data are available since the opening of the French high-speed lines.

Usually, the synthesized Mauzin indicator of the longitudinal levelling (NL indicator) is used for maintenance issues: thresholds are fixed for this indicator in order to obtain a classification of the track condition and to fix dates for maintenance operations. For example, an intervention should be scheduled before the NL indicator exceeds 0.9.

Based on expert judgements, a Gamma process has been used in [1] both to model the evolution of the NL indicator and to plan maintenance actions. As noted by J.M. van Noortwijk in his recent survey [2], this kind of process is widely used in reliability studies (see also [3], [4] and [5]). Various domains of applications exist, such as civil engineering ([6], [7]), highway engineering [8] or railway engineering [9]. Gamma processes are also used in other domains, such as finance [10] or risk analysis [11]. All these papers use univariate Gamma processes.

In the present case, as the two indicators NL and NT are dependent, the use of a bivariate model is required. For this purpose, different processes might be used, such as Bessel [12] or Lévy processes [13]. In this paper, the approach of F.A. Buijs, J.W. Hall, J.M. van Noortwijk and P.B. Sayers in [6] is used: a specific Lévy process called bivariate Gamma process is considered. This process is constructed from three independent univariate Gamma processes by trivariate reduction, and has univariate Gamma processes as marginal processes.

It is the first time that both NL and NT indicators are used conjointly to predict the optimal dates of maintenance actions. The objective is to analyse the correlation between the two processes and to determine in what circumstances this bivariate process allows a better prediction of the maintenance times than the current univariate one, based only on the NL indicator.

The paper is organized in the following way: bivariate Gamma processes are introduced in Section 2. Empirical and maximum likelihood estimators for their parameters are provided in Section 3. An EM algorithm is proposed to carry out the maximum likelihood estimation. Both methods are tested on simulated data. Section 4 is devoted to the study of preventive maintenance planning and to the comparison of the results based on the bivariate and on the univariate models. Finally, a bivariate Gamma process is fitted to real data of railway track deterioration in Section 5 and it is shown that the preventive maintenance scheduling based on the two available deterioration indicators are clearly safer than those based on a single one, or on both taken separately.

2 THE BIVARIATE GAMMA PROCESS

Recall that an univariate (homogeneous) Gamma process $(Y_t)_{t\geq 0}$ with parameters $(\alpha, b) \in \mathbb{R}_+^{*2}$ is a process with independent increments such that Y_t is Gamma distributed $\Gamma(\alpha t, b)$ with probability density function (p.d.f.)

$$f_{\alpha t,b}(x) = \frac{b^{\alpha t}}{\Gamma(\alpha t)} x^{\alpha t-1} e^{-bx} \mathbf{1}_{\mathbb{R}_+}(x),$$

$\mathbb{E}(Y_t) = \frac{\alpha t}{b}$, $\mathrm{Var}(Y_t) = \frac{\alpha t}{b^2}$ for all $t > 0$, and $Y_0 \equiv 0$ (see [2] for more details).

Following [6], a bivariate Gamma process $\left(X_t\right)_{t\geq 0} = \left(X_t^{(1)}, X_t^{(2)}\right)_{t\geq 0}$ is constructed by trivariate reduction: starting from three independent univariate Gamma processes $\left(Y_t^{(i)}\right)_{t\geq 0}$ with parameters $(\alpha_i, 1)$ for $i \in \{1, 2, 3\}$ and from $b_1 > 0$, $b_2 > 0$, one defines:

$$X_t^{(1)} = \left(Y_t^{(1)} + Y_t^{(3)}\right)/b_1, \text{ and } X_t^{(2)} = \left(Y_t^{(2)} + Y_t^{(3)}\right)/b_2 \quad \text{for all} \quad t \geq 0.$$

The process $(X_t)_{t\geq 0} = \left(X_t^{(1)}, X_t^{(2)}\right)_{t\geq 0}$ is then a homogeneous process in time with independent increments and it is a Lévy process. The marginal processes of $(X_t)_{t\geq 0}$ are univariate Gamma processes with respective parameters (a_i, b_i), where $a_i = \alpha_i + \alpha_3$ for $i = 1, 2$.

For any bivariate Lévy process, the correlation coefficient ρ_{X_t} of $X_t^{(1)}$ and $X_t^{(2)}$ is known to be independent of t. For a bivariate Gamma process, one obtains:

$$\rho = \rho_{X_t} = \frac{\alpha_3}{\sqrt{a_1 a_2}}$$

and

$$\alpha_1 = a_1 - \rho\sqrt{a_1 a_2}, \quad \alpha_2 = a_2 - \rho\sqrt{a_1 a_2}, \quad \alpha_3 = \rho\sqrt{a_1 a_2}.$$

This entails

$$0 \leq \rho \leq \rho_{\max} = \frac{\min(a_1, a_2)}{\sqrt{a_1 a_2}}. \tag{1}$$

See [14] section XI.3 for results on bivariate Gamma distributions.

This leads to two equivalent parameterizations of a bivariate Gamma process: $(\alpha_1, \alpha_2, \alpha_3, b_1, b_2)$ and $(a_1, a_2, b_1, b_2, \rho)$.

With the parameterization $(\alpha_1, \alpha_2, \alpha_3, b_1, b_2)$, the joint p.d.f. of X_t is:

$$\begin{aligned}
&g_t(x_1, x_2) \\
&= b_1 b_2 \int_0^{\min(b_1x_1, b_2x_2)} f_{\alpha_1 t,1}(b_1x_1 - x_3)\, f_{\alpha_2 t,1}(b_2x_2 - x_3)\, f_{\alpha_3 t,1}(x_3)\; dx_3, \\
&= \frac{b_1 b_2 e^{-b_1x_1 - b_2x_2}}{\Gamma(\alpha_1 t)\,\Gamma(\alpha_2 t)\,\Gamma(\alpha_3 t)} \times \cdots \\
&\times \int_0^{\min(b_1x_1, b_2x_2)} (b_1x_1 - x_3)^{\alpha_1 t - 1} (b_2x_2 - x_3)^{\alpha_2 t - 1} x_3^{\alpha_3 t - 1} e^{-x_3}\; dx_3. \qquad (2)
\end{aligned}$$

3 PARAMETER ESTIMATION

The data used for the parameter estimation are values of the process increments for non overlapping time intervals on a single trajectory, and also on different independent trajectories. The data can then be represented as $\left(\Delta t_j, \Delta X_j^{(1)}(\omega), \Delta X_j^{(2)}(\omega)\right)_{1\le j\le n}$ where $\Delta t_j = t_j - s_j$ stands for a time increment and $\Delta X_j^{(i)} = X_{t_j}^{(i)} - X_{s_j}^{(i)}$ for the associated i-th marginal increment $(i = 1, 2)$. For different j, the random vectors $\left(\Delta X_j^{(1)}, \Delta X_j^{(2)}\right)$ are independent, but not identically distributed. The random variable $\Delta X_j^{(i)}$ $(i = 1, 2)$ is Gamma distributed with parameters $(a_i\ \Delta t_j, b_i)$. The joint p.d.f. of the random vector $\left(\Delta X_j^{(1)}, \Delta X_j^{(2)}\right)$ is equal to $g_{\Delta t_j}(.,.)$, with Δt_j substituted for t in (2). In the same way as for parameter estimation of a (univariate) Gamma process, both empirical and maximum likelihood methods are possible in the bivariate case.

3.1 Empirical estimators

Using $\mathbb{E}\left(\Delta X_j^{(i)}\right) = \frac{a_i}{b_i}\,\Delta t_j$ and $\mathrm{Var}\left(\Delta X_j^{(i)}\right) = \frac{a_i}{b_i^2}\,\Delta t_j$ for $i = 1, 2$ and for all j, empirical estimators $(\hat{a}_1, \hat{b}_1, \hat{a}_2, \hat{b}_2)$ of (a_1, b_1, a_2, b_2) are given in [7] and [15], with:

$$\frac{\hat{a}_i}{\hat{b}_i} = \frac{\sum_{j=1}^n \Delta X_j^{(i)}}{t_n} \quad \text{and} \quad \frac{\hat{a}_i}{\hat{b}_i^2} = \frac{\sum_{j=1}^n \left(\Delta X_j^{(i)} - \frac{\hat{a}_i}{\hat{b}_i}\Delta t_j\right)^2}{t_n - \frac{1}{t_n}\sum_{j=1}^n (\Delta t_j)^2}, \qquad (3)$$

where we set $t_n = \sum_{j=1}^n \Delta t_j$.

Using

$$\mathrm{Cov}\left(\Delta X_j^{(1)}, \Delta X_j^{(2)}\right) = \rho \frac{\sqrt{a_1 a_2}}{b_1 b_2} \Delta t_j,$$

a similar estimator $\hat{\rho}$ may be given for ρ, with:

$$\hat{\rho}\frac{\sqrt{\hat{a}_1\hat{a}_2}}{\hat{b}_1\hat{b}_2}=\frac{\sum_{j=1}^{n}\left(\Delta X_j^{(1)}-\frac{\hat{a}_1}{\hat{b}_1}\Delta t_j\right)\left(\Delta X_j^{(2)}-\frac{\hat{a}_2}{\hat{b}_2}\Delta t_j\right)}{t_n-\frac{1}{t_n}\sum_{j=1}^{n}\left(\Delta t_j\right)^2.} \tag{4}$$

These estimators satisfy:

$$\mathbb{E}\left(\frac{\hat{a}_i}{\hat{b}_i}\right)=\frac{a_i}{b_i},\quad \mathbb{E}\left(\frac{\hat{a}_i}{\hat{b}_i^2}\right)=\frac{a_i}{b_i^2},\quad \mathbb{E}\left(\hat{\rho}\frac{\sqrt{\hat{a}_1\hat{a}_2}}{\hat{b}_1\hat{b}_2}\right)=\rho\frac{\sqrt{a_1a_2}}{b_1b_2}.$$

If the time increments Δt_j are equal, these estimators coïncides with the usual empirical estimators in the case of i.i.d. random variables.

3.2 Maximum likelihood estimators

The parameter estimation of an univariate Gamma process is usually done by maximizing the likelihood function (see e.g. [1]). With this method, estimators $\bar{a}_i$ and $\bar{b}_i$ $(i=1,2)$ of the marginal parameters are computed by solving the equations:

$$\frac{\bar{a}_i}{\bar{b}_i}=\frac{\sum_{j=1}^{n}\Delta X_j^{(i)}}{\sum_{j=1}^{n}\Delta t_j}\quad\text{and}$$

$$\left(\sum_{j=1}^{n}\Delta t_j\right)\times\ln\left(\bar{a}_i\frac{\sum_{j=1}^{n}\Delta t_j}{\sum_{j=1}^{n}\Delta X_j^{(i)}}\right)+\sum_{j=1}^{n}\Delta t_j\left[\ln\left(\Delta X_j^{(i)}\right)-\psi\left(\bar{a}_i\ \Delta t_j\right)\right]=0,$$

where

$$\psi\left(x\right)=\frac{\Gamma'\left(x\right)}{\Gamma\left(x\right)},\ \Gamma\left(x\right)=\int_0^{\infty}e^{-u}u^{x-1}du$$

for all $x>0$ (ψ is the Digamma function).

In order to estimate all the parameters of the bivariate process $(\alpha_1,\alpha_2,\alpha_3,b_1,b_2)$ (which are here prefered to (a_1,b_1,a_2,b_2,ρ)), the likelihood function associated with the data $\left(\Delta t_j,\Delta X_j^{(1)},\Delta X_j^{(2)}\right)_{1\leq j\leq n}$ can be written as $\mathcal{L}(\alpha_1,\alpha_2,\alpha_3,b_1,b_2)=\prod_{j=1}^{n}g_{\Delta t_j}(\Delta X_j^{(1)},\Delta X_j^{(2)})$. However, because of the expression of the function $g_t(.,.)$, it seems complicated to optimize this likelihood function directly. An EM algorithm (see [16]) is then used, considering $\left(\Delta Y_j^{(3)}=Y_{t_j}^{(3)}-Y_{s_j}^{(3)}\right)_{1\leq j\leq n}$ as hidden data. This procedure is still too complicated to estimate all the five parameters and does not work numerically. So, the procedure is restricted to the three parameters $(\alpha_1,\alpha_2,\alpha_3)$. For the parameters b_1,b_2, the values $\left(\bar{b}_1,\bar{b}_2\right)$ computed using the maximum likelihood method for each univariate marginal process are taken.

In order to simplify the expressions, the values of the data $\left(\Delta t_j,\ \Delta X_j^{(1)},\ \Delta X_j^{(2)},\ \Delta Y_j^{(3)}\right)_{1\leq j\leq n}$ are denoted by $\left(t_j,x_j^{(1)},x_j^{(2)},y_j^{(3)}\right)_{1\leq j\leq n}$, the associated n-dimensional random vectors by $\left(\overline{X}^{(1)},\overline{X}^{(2)},\overline{Y}^{(3)}\right)$ and the associated n-dimensional data vectors by $\left(\overline{x}^{(1)},\overline{x}^{(2)},\overline{y}^{(3)}\right)$.

The joint p.d.f. of the random vector $\left(X_t^{(1)}, X_t^{(2)}, Y_t^{(3)}\right)$ is equal to:

$$b_1 b_2 f_{\alpha_1 t,1}\left(b_1 x_1 - y_3\right) f_{\alpha_2 t,1}\left(b_2 x_2 - y_3\right) f_{\alpha_3 t,1}\left(y_3\right) = \frac{b_1 b_2 e^{-(b_1 x_1 + b_2 x_2)}}{\Gamma\left(\alpha_1 t\right)\Gamma\left(\alpha_2 t\right)\Gamma\left(\alpha_3 t\right)}\left(b_1 x_1 - y_3\right)^{\alpha_1 t - 1}\left(b_2 x_2 - y_3\right)^{\alpha_2 t - 1} y_3^{\alpha_3 t - 1} e^{y_3},$$

with $0 \leq y_3 \leq \min\left(b_1 x_1, b_2 x_2\right)$, $x_1 > 0$ and $x_2 > 0$. Then, the log-likelihood function $Q\big(\bar{x}^{(1)}, \bar{x}^{(2)}, \bar{y}^{(3)}\big)$ associated with the complete data $\big(\bar{x}^{(1)}, \bar{x}^{(2)}, \bar{y}^{(3)}\big)$ is derived:

$$\begin{aligned}
Q\big(\bar{x}^{(1)}, \bar{x}^{(2)}, \bar{y}^{(3)}\big) &= n\left(\ln\left(b_1\right) + \ln\left(b_2\right)\right) - \cdots \\
&\sum_{j=1}^{n}\left(\ln\Gamma\left(\alpha_1 t_j\right) + \ln\Gamma\left(\alpha_2 t_j\right) + \ln\Gamma\left(\alpha_3 t_j\right)\right) - b_1 \sum_{j=1}^{n} x_j^{(1)} - \cdots \\
&\quad b_2 \sum_{j=1}^{n} x_j^{(2)} + \sum_{j=1}^{n}\Big\{\left(\alpha_1 t_j - 1\right)\ln\big(b_1 x_j^{(1)} - y_j^{(3)}\big) + \cdots \\
&\quad \left(\alpha_2 t_j - 1\right)\ln\left(b_2 x_j^{(2)} - y_j^{(3)}\right) + \left(\alpha_3 t_j - 1\right)\ln\left(y_j^{(3)}\right) + y_j^{(3)}\Big\}.
\end{aligned}$$

For the EM algorithm, the conditional log-likelihood of the complete data given the observed data is needed:

$$\begin{aligned}
&\mathbb{E}\big(Q\big(\bar{X}^{(1)}, \bar{X}^{(2)}, \bar{Y}^{(3)}\big) | \bar{X}^{(1)} = \bar{x}^{(1)}, \bar{X}^{(2)} = \bar{x}^{(2)}\big) \\
&= n\left(\ln\left(b_1\right) + \ln\left(b_2\right)\right) - b_1 \sum_{j=1}^{n} x_j^{(1)} - b_2 \sum_{j=1}^{n} x_j^{(2)} + \cdots \\
&\sum_{j=1}^{n}\ \Big\{\big(\left(\alpha_1 t_j - 1\right)\mathbb{E}\big(\ln\big(b_1 x_j^{(1)} - \Delta Y_j^{(3)}\big) | \Delta X_j^{(1)} = x_j^{(1)}, \Delta X_j^{(2)} = x_j^{(2)}\big) \\
&\quad + \left(\alpha_2 t_j - 1\right)\mathbb{E}\big(\ln\big(b_2 x_j^{(2)} - \Delta Y_j^{(3)}\big) | \Delta X_j^{(1)} = x_j^{(1)}, \Delta X_j^{(2)} = x_j^{(2)}\big) \\
&\quad + \left(\alpha_3 t_j - 1\right)\mathbb{E}\big(\ln\big(\Delta Y_j^{(3)}\big) | \Delta X_j^{(1)} = x_j^{(1)}, \Delta X_j^{(2)} = x_j^{(2)}\big) \\
&\quad + \mathbb{E}\big(Y_j^{(3)} | \Delta X_j^{(1)} = x_j^{(1)}, \Delta X_j^{(2)} = x_j^{(2)}\big)\Big\} \\
&- \sum_{j=1}^{n}\left(\ln\Gamma\left(\alpha_1 t_j\right) + \ln\Gamma\left(\alpha_2 t_j\right) + \ln\Gamma\left(\alpha_3 t_j\right)\right).
\end{aligned} \tag{5}$$

Finally, the conditional density function of $Y_t^{(3)}$ given $X_t^{(1)} = x_1, X_t^{(2)} = x_2$ is equal to:

$$\begin{aligned}
&\frac{f_{\alpha_1 t,1}\left(b_1 x_1 - y_3\right) f_{\alpha_2 t,1}\left(b_2 x_2 - y_3\right) f_{\alpha_3 t,1}\left(y_3\right)}{\int_0^{\min(b_1 x_1, b_2 x_2)} f_{\alpha_1 t,1}\left(b_1 x_1 - x_3\right) f_{\alpha_2 t,1}\left(b_2 x_2 - x_3\right) f_{\alpha_3 t,1}\left(x_3\right)\ dx_3} \\
&= \frac{\left(b_1 x_1 - y_3\right)^{\alpha_1 t - 1}\left(b_2 x_2 - y_3\right)^{\alpha_2 t - 1} y_3^{\alpha_3 t - 1} e^{y_3}}{\int_0^{\min(b_1 x_1, b_2 x_2)}\left(b_1 x_1 - x_3\right)^{\alpha_1 t - 1}\left(b_2 x_2 - x_3\right)^{\alpha_2 t - 1} x_3^{\alpha_3 t - 1} e^{x_3}\ dx_3},
\end{aligned}$$

where $0 \leq y_3 \leq \min(b_1 x_1, b_2 x_2)$, $x_1 > 0$ and $x_2 > 0$.

Step k of the EM algorithm consists of computing new parameter values $(\alpha_1^{(k+1)}, \alpha_2^{(k+1)}, \alpha_3^{(k+1)})$ given the current values $(\alpha_1^{(k)}, \alpha_2^{(k)}, \alpha_3^{(k)})$ in two stages:

- stage 1: compute the conditional expectations in (5) using the current set $(\alpha_1^{(k)}, \alpha_2^{(k)}, \alpha_3^{(k)})$ of parameters, with:

$$\begin{aligned}
f_1\left(j, \alpha_1^{(k)}, \alpha_2^{(k)}, \alpha_3^{(k)}\right) &= \mathbb{E}\left(\ln\left(\bar{b}_1 \bar{x}_j^{(1)} - \bar{Y}_j^{(3)}\right) | \bar{X}^{(1)} = \bar{x}_j^{(1)}, \bar{X}^{(2)} = \bar{x}_j^{(2)}\right), \\
f_2\left(j, \alpha_1^{(k)}, \alpha_2^{(k)}, \alpha_3^{(k)}\right) &= \mathbb{E}\left(\ln\left(\bar{b}_2 \bar{x}_j^{(2)} - \bar{Y}_j^{(3)}\right) | \bar{X}^{(1)} = \bar{x}_j^{(1)}, \bar{X}^{(2)} = \bar{x}_j^{(2)}\right), \\
f_3\left(j, \alpha_1^{(k)}, \alpha_2^{(k)}, \alpha_3^{(k)}\right) &= \mathbb{E}\left(\ln\left(\bar{Y}_j^{(3)}\right) | \bar{X}^{(1)} = \bar{x}_j^{(1)}, \bar{X}^{(2)} = \bar{x}_j^{(2)}\right), \\
h\left(\alpha_1^{(k)}, \alpha_2^{(k)}, \alpha_3^{(k)}\right) &= \sum_{j=1}^{n} \mathbb{E}\left(\bar{Y}_j^{(3)} | \bar{X}^{(1)} = \bar{x}_j^{(1)}, \bar{X}^{(2)} = \bar{x}_j^{(2)}\right).
\end{aligned}$$

- stage 2: take for $(\alpha_1^{(k+1)}, \alpha_2^{(k+1)}, \alpha_3^{(k+1)})$ the values of $(\alpha_1, \alpha_2, \alpha_3)$ that maximize (5), which here becomes:

$$\begin{aligned}
& g\big(\alpha_1, \alpha_2, \alpha_3, \alpha_1^{(k)}, \alpha_2^{(k)}, \alpha_3^{(k)}\big) \\
&= n\left(\ln\left(\bar{b}_1\right) + \ln\left(\bar{b}_2\right)\right) - \bar{b}_1 \sum_{j=1}^{n} x_j^{(1)} - \bar{b}_2 \sum_{j=1}^{n} x_j^{(2)} \\
&+ \sum_{j=1}^{n} \Big\{ (\alpha_1 t_j - 1) f_1\big(j, \alpha_1^{(k)}, \alpha_2^{(k)}, \alpha_3^{(k)}\big) + (\alpha_2 t_j - 1) f_2\big(j, \alpha_1^{(k)}, \alpha_2^{(k)}, \alpha_3^{(k)}\big) \\
&\qquad + (\alpha_3 t_j - 1) f_3\big(j, \alpha_1^{(k)}, \alpha_2^{(k)}, \alpha_3^{(k)}\big) \Big\} \\
&- \sum_{j=1}^{n} \left(\ln\Gamma(\alpha_1 t_j) + \ln\Gamma(\alpha_2 t_j) + \ln\Gamma(\alpha_3 t_j)\right) + h\big(\alpha_1^{(k)}, \alpha_2^{(k)}, \alpha_3^{(k)}\big).
\end{aligned}$$

The maximization in stage 2 is done by solving the following equation with respect to α_i:

$$\frac{\partial g\big(\alpha_1, \alpha_2, \alpha_3, \alpha_1^{(k)}, \alpha_2^{(k)}, \alpha_3^{(k)}\big)}{\partial \alpha_i} = \sum_{j=1}^{n} t_j f_i\big(j, \alpha_1^{(k)}, \alpha_2^{(k)}, \alpha_3^{(k)}\big) - \sum_{j=1}^{n} t_j \psi(\alpha_i t_j) = 0 \quad (6)$$

for $i = 1, 2, 3$.

In the same way, it is possible to take the values $\left(\bar{a}_1, \bar{a}_2, \bar{b}_1, \bar{b}_2\right)$ obtained by maximum likelihood estimation on the univariate marginal processes for (a_1, a_2, b_1, b_2) and to estimate only the last parameter α_3 by the EM algorithm. In that case, $\alpha_3^{(k+1)}$ is the solution of the equation:

$$\sum_{j=1}^{n} t_j \Big\{ f_3\big(j, \alpha_1^{(k)}, \alpha_2^{(k)}, \alpha_3^{(k)}\big) - f_1\big(j, \alpha_1^{(k)}, \alpha_2^{(k)}, \alpha_3^{(k)}\big) - \cdots$$
$$f_2\big(j, \alpha_1^{(k)}, \alpha_2^{(k)}, \alpha_3^{(k)}\big) \Big\} - \sum_{j=1}^{n} t_j \Big\{ \psi\left(\alpha_3 t_j\right) - \psi\left((\bar{a}_1 - \alpha_3) t_j\right) - \cdots$$
$$\psi\left((\bar{a}_2 - \alpha_3) t_j\right) \Big\} = 0.$$

3.3 Tests on simulated data

500 time increments $(t_j)_{1 \leq j \leq 500}$ are randomly chosen similar to the data of track deterioration (the proposed methods will be used on these data in Section 5). Then, 500 values of a bivariate Gamma process are simulated corresponding to these time increments and with parameters $a_1 = 0.33, a_2 = 0.035, b_1 = 13.5, b_2 = 20$ and $\rho = 0.5296$. These parameter values have the same order of magnitude than those observed for track deterioration studied in Section 5. Three series of 500 data points are simulated independently. Results of parameters estimation are given in Tables 1, 2 and 3, each corresponding to a series of data. In these tables, one can find: the true values in column 2, the empirical estimators in column 3, the univariate maximum likelihood estimators of a_1, b_1, a_2, b_2 in column 4, the EM estimator of the three parameters a_1, a_2, ρ in column 5, using the parameters $\bar{b}_1, \bar{b}_2$ previously estimated by the univariate maximum likelihood method (from column 4), and the second EM estimator of the parameter ρ in column 6, using the estimated parameters $\bar{a}_1, \bar{b}_1, \bar{a}_2, \bar{b}_2$ from column 4.

The initial values for the EM algorithm are different for the three tables. For Table 1, the EM algorithm has been initiated with $\alpha_1^{(0)} = \alpha_2^{(0)} = 0.05$ and $\alpha_3^{(0)} = 0.15$ ($a_1^{(0)} = a_2^{(0)} = 0.1$ and $\rho^{(0)} = 0.75$). For Tables 2 and 3, $\alpha_1^{(0)} = \alpha_2^{(0)} = \alpha_3^{(0)} = 0.01$, and $\alpha_1^{(0)} = 0.02$, $\alpha_2^{(0)} = 0.01$, $\alpha_3^{(0)} = 0.05$ were respectively taken.

Looking at the development of $a_i^{(k)}$ and $\rho^{(k)}$ along the different steps of the EM algorithm, one may note that the parameters $a_i^{(k)}$ stabilize more quickly than the parameter $\rho^{(k)}$ (about 5 iterations for $a_i^{(k)}$ and between 20 and 30 iterations for $\rho^{(k)}$).

The conclusion of this section is that estimation of parameters (a_i, b_i) by empirical and maximum likelihood methods give satisfactory results, with a slight preference to maximum likelihood. Empirical estimators of ρ have a good order of magnitude, but are sometimes not precise enough. Estimators of ρ obtained by EM are always reasonable. The estimation of the three parameters $(\alpha_1, \alpha_2, \alpha_3)$ (column EM1) seems to give slightly better results

	True values	Empirical estimators	Univariate max likelihood	EM algorithm EM1	EM2
a_1	0.0330	0.0348	0.0342	0.0347	–
b_1	13.5	14.38	14.14	–	–
a_2	0.0350	0.0362	0.0357	0.0354	–
b_2	20	20.58	20.25	–	–
ρ	0.5296	0.5637	–	0.5231	0.5214

TABLE 1. Results for the first series of data.

	True values	Empirical estimators	Univariate max likelihood	EM algorithm EM1	EM2
a_1	0.0330	0.0315	0.0326	0.0328	–
b_1	13.5	12.80	13.16	–	–
a_2	0.0350	0.0357	0.0361	0.0365	–
b_2	20	20.25	20.54	–	–
ρ	0.5296	0.5750	–	0.5272	0.5257

TABLE 2. Results for the second series of data.

than those of the estimation of the parameter α_3 alone (column EM2). The results obtained by the EM algorithm for parameters a_i (column EM1) are good, with a quality quite similar to those obtained by univariate maximum likelihood estimation. Finally, the EM algorithm does not seem sensitive to initial values, at least if the initial value of α_3 is not too small.

4 PREVENTIVE MAINTENANCE PLANNING

A bivariate Gamma process $X_t = \left(X_t^{(1)}, X_t^{(2)}\right)$ is now used to model the development of two deterioration indicators of a system. We assume that there exist (corrective) thresholds s_i $(i = 1, 2)$ for each indicator, above which the system is considered to be deteriorated. The system is not continuously monitored but only inspected at time intervals, with a perfect observation of the deterioration level. When one (or both) indicator(s) is observed to be beyond its corrective threshold, an instantaneous maintenance action is undertaken, which brings the system back to a better state, not necessarily as good as new. When both indicators are observed to be below their corrective thresholds or after a maintenance action, a new inspection is planned. The time to next inspection (τ) must ensure with a high probability that neither $X_t^{(1)}$ nor $X_t^{(2)}$ go beyond their corrective thresholds s_i before the next inspection.

Let $(x_1, x_2) \in [0, s_1[\times[0, s_2[$ be the observed deterioration level at some inspection time, say at time $t = 0$ with no restriction. (If $x_1 \geq s_1$ or $x_2 \geq s_2$, a maintenance action is immediately undertaken).

For $i = 1, 2$, let $T^{(i)}$ be the hitting time of the threshold s_i for the marginal process $\left(X_t^{(i)}\right)_{t\geq 0}$, with $T^{(i)} = \inf\left(t > 0 : X_t^{(i)} \geq s_i\right)$. Also, let $\varepsilon \in]0, 1[$ be some confidence level.

Different points of view are possible: in the first case, $\tau^{(i)}$, $i = 1, 2$ is the

	True values	Empirical estimators	Univariate max likelihood	EM algorithm EM1	EM2
a_1	0.0330	0.0297	0.0340	0.0343	–
b_1	13.5	11.71	13.43	–	–
a_2	0.0350	0.0340	0.0385	0.0389	–
b_2	20	18.79	21.28	–	–
ρ	0.5296	0.5645	–	0.5060	0.5027

TABLE 3. Results for the third series of data.

	a_2	b_2	x_1	x_2	$\rho_{\max}$	$\tau^{(1)}$	$\tau^{(2)}$	τ^{U}	$\tau^{B}(\rho_{\max})$
case 1	0.03	30	0.2	0.2	1	341.12	558.31	341.12	341.12
case 2	0.04	20	0.4	0.2	0.866	237.33	255.84	237.33	229.91

TABLE 4. Two different combinations of values for a_2, b_2, x_1 and x_2, and the resulting $\rho_{\max}$, $\tau^{(1)}$, $\tau^{(2)}$, τ^U and $\tau^B(\rho_{\max})$.

time to next inspection associated to the marginal process $\left(X_t^{(i)}\right)_{t\geq 0}$, with

$$\tau^{(i)} = \max\bigl(\tau \geq 0 \text{ such that } \mathbb{P}_{x_i}\bigl(T^{(i)} > \tau\bigr) \geq 1-\varepsilon\bigr),$$

where $\mathbb{P}_{x_i}$ stands for the conditional probability given $X_0^{(i)} = x_i$. One then gets: $\mathbb{P}_{x_i}\left(T^{(i)} > \tau^{(i)}\right) = 1-\varepsilon$.

Without a bivariate model, a natural time to next inspection for the system is:

$$\begin{aligned}\tau^U &= \max\bigl(\tau \geq 0 \text{ s.t. } \mathbb{P}_{x_1}\bigl(T^{(1)} > \tau\bigr) \geq 1-\varepsilon \text{ and } \mathbb{P}_{x_2}\bigl(T^{(2)} > \tau\bigr) \geq 1-\varepsilon\bigr),\\ &= \min\bigl(\tau^{(1)}, \tau^{(2)}\bigr).\end{aligned}$$

Using a bivariate Gamma process, the time to next inspection becomes:

$$\tau^B = \max\bigl(\tau \geq 0 \text{ such that } \mathbb{P}_{(x_1,x_2)}\bigl(T^{(1)} > \tau, T^{(2)} > \tau\bigr) \geq 1-\varepsilon\bigr).$$

The goal is to compare τ^U and τ^B, and more generally, to understand the influence of the dependence between both components on τ^B. Using

$$\mathbb{P}_{x_i}\bigl(T^{(i)} > t\bigr) = \mathbb{P}_{x_i}\bigl(X_t^{(i)} < s_i\bigr) = \mathbb{P}_0\bigl(X_t^{(i)} < s_i - x_i\bigr) = F_{a_i t, b_i}\left(s_i - x_i\right),$$

where $F_{a_i t, b_i}(x)$ is the cumulative distribution function of the distribution $\Gamma(a_i t, b_i)$, the real $\tau^{(i)}$ is computed by solving the equation $F_{a_i \tau^{(i)}, b_i}(s_i - x_i) = 1-\varepsilon$, for $i = 1, 2$, and $\tau^U = \min\left(\tau^{(1)}, \tau^{(2)}\right)$ is derived. Similarly,

$$\begin{aligned}\mathbb{P}_{(x_1,x_2)}\bigl(T^{(1)} > t, T^{(2)} > t\bigr) &= \mathbb{P}_{(0,0)}\bigl(X_t^{(1)} < s_1 - x_1, X_t^{(2)} < s_2 - x_2\bigr),\\ &= \int_0^{s_1-x_1}\int_0^{s_2-x_2} g_t\left(y_1, y_2\right)\, dy_1\, dy_2,\\ &\equiv G_t\left(s_1 - x_1, s_2 - x_2\right),\end{aligned}$$

where g_t is the p.d.f. of X_t (see (2)). This provides τ^B by solving $G_{\tau^B}(s_1 - x_1, s_2 - x_2) = 1-\varepsilon$.

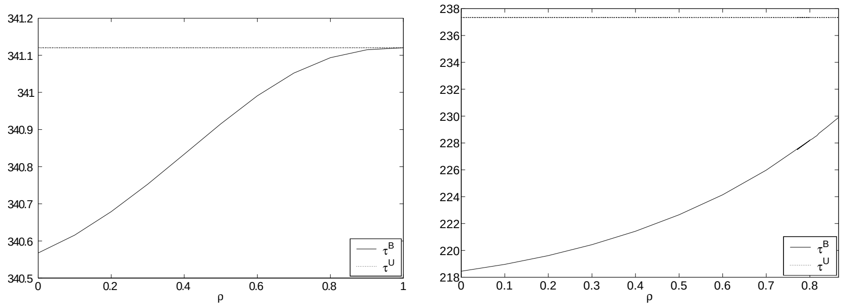

FIGURE 2. τ^B with respect to ρ and τ^U, for the four cases of Table 4

With $a_1 = 0.03$, $b_1 = 20$, $\epsilon = 0.5$ and $s_1 = s_2 = 1$, and different values for a_2, b_2, x_1 and x_2, Table 4 gives the corresponding values for $\rho_{\max}$ (as provided by (1)) and the resulting $\tau^{(1)}$, $\tau^{(2)}$, τ^U and τ^B $(\rho_{\max})$. The value of τ^B is plotted with respect to ρ in the Figures 2 for the two different cases of Table 4, and the corresponding value of τ^U is indicated.

In both figures, one can observe that with all other parameters fixed, the bivariate preventive time τ^B is an increasing function of ρ, such that $\tau^B \leq \tau^U$. Also, both $\tau^B = \tau^U$ and $\tau^B < \tau^U$ are possible. The theoretical proof of such results is not provided here because of the reduced size of the present paper, but will be provided in a forthcomming one.

In conclusion to this section, one can see that using a bivariate model instead of two separate univariate models generally shortens the time to next inspection ($\tau^B \leq \tau^U$). This means that taking into account the dependence between both components provides safer results. Also, the optimal time to next inspection is increasing with dependence (τ^B increases with ρ), which implies that the error made when considering separate models (τ^U) is all the more important the less the components are dependent. This also implies that the safest attitude, in case of an unkown correlation, is to consider both components as independent and chose $\tau = \tau^{\perp}$, where

$$\tau^{\perp} = \max\big(\tau \geq 0 \text{ such that } \mathbb{P}_{x_1}\big(T^{(1)} > \tau\big)\mathbb{P}_{x_2}\big(T^{(2)} > \tau\big) \geq 1 - \varepsilon\big).$$

5 APPLICATION TO TRACK MAINTENANCE

A bivariate Gamma process is now used to model the evolution of the two track indicators NL and NT (see the Introduction) and times to next inspection are computed, as described in the previous section.

Using univariate maximum likelihood and EM methods on data corresponding to the Paris-Lyon high-speed line provide the estimations $\hat{a}_1 = 0.0355, \hat{b}_1 = 19.19, \hat{a}_2 = 0.0387, \hat{b}_2 = 29.72, \hat{\rho} = 0.5262$. Usual thresholds are

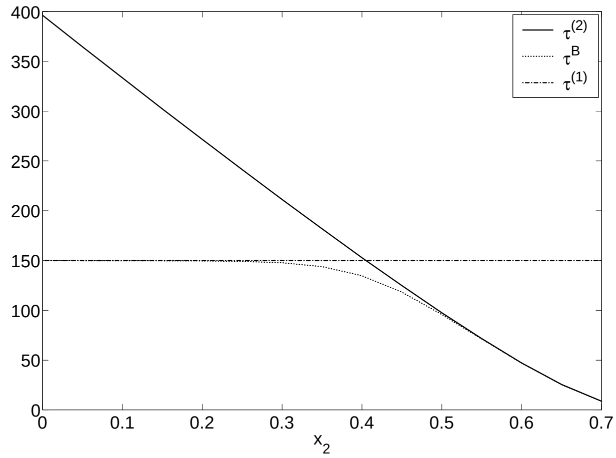

FIGURE 3. $\tau^{(1)}$, $\tau^{(2)}$ and τ^B with respect to x_2 with $x_1 = 0.4$

$s_1 = 0.9$ for NL and $s_2 = 0.75$ for NT. With these values, $\tau^{(1)}$, $\tau^{(2)}$ and τ^B are plotted in Figure 3 with respect of x_2 when x_1 is fixed ($x_1 = 0.4$). In that case $\tau^{(1)} = 150$.

This figure shows that taking into account the single information $x_1 = 0.4$ may lead to too late maintenance actions. As an example, if $x_2 = 0.4$, one has $\tau^B = 134.7$ (and $\tau^{(2)} = 152.9$). The preventive maintenance action based only on NL is consequently scheduled 15 days too lately. If $x_2 = 0.5$, one obtains $\tau^B = 95.9$ ($\tau^{(2)} = 97.5$) and the maintenance action is undertaken 54 days too late. If $x_2 = 0.6$, one obtains $\tau^B = 47.1$ ($\tau^{(2)}= 47.2$) and this is 103 days too late.

Concluding this section, one can finally observe that if x_1 is not too close to x_2, the value $\tau^U = \min\left(\tau^{(1)}, \tau^{(2)}\right)$ seams reasonable for maintenance scheduling (see Figure 3), contrary to the currently used $\tau^{(1)}$, which may entail large delay in its planning (more than 100 days in our example). If x_1 is close to x_2, the values of τ^U and τ^B have the same order of magnitude, however with $\tau^U > \tau^B$, so that the preventive maintenance action is again planned too lately (15 days in the example).

6 CONCLUSION

A bivariate Gamma process has been used to model the development of two deterioration indicators. Different estimation methods have been proposed for the parameters and tested on simulated data. Based on these tests, the best estimators seem provided by univariate likelihood maximization for the marginal parameters and by an EM algorithm for the correlation coefficient.

Preventive maintenance scheduling has then been studied for a system that deteriorates according to a bivariate Gamma process. In particular, it has been shown that, given an observed bivariate deterioration level, the optimal time to maintenance is increasing with dependence. It has been proven that the optimal time to maintenance is always shorter when taking into account the dependence between the two deterioration indicators than when considering them separately (or only considering one of them).

Finally, a bivariate Gamma process has been used to study a real track maintenance problem. The application shows that when both observed deterioration indicators are close to each other, the bivariate process gives safer results for maintenance scheduling than both univariate processes considered separately or one single univariate process, with the same order of magnitude in each case however. When the observed deterioration indicators are clearly different, considering one single univariate process as it is done in current track maintenance, may lead to clearly unadaquate results. This application to real data of railway track deterioration hence shows the interest of a bivariate model for a correct definition of a maintenance strategy.

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